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Robust Estimation Algorithm for Both Switching Signal and State of Switched Linear Systems

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Abstract: We present a hybrid-type observer for detecting the switching time and estimating both the active mode and the states of continuous-time switched linear systems. The systems under consideration have external inputs and are affected by unknown disturbances. In addition, noise corrupts the output measurements. In this setting the switching cannot be detected immediately, and thus, this paper presents a condition that relates the amount of delay to the sizes of the unknown disturbances/noises, the external inputs, and the states, and the strength of the observability. Once the condition is satisfied, the proposed observer and algorithm return the exact active mode and approximate state information of the switched system. A numerical example is also presented to show the performance of our algorithm..

Keywords: Switched linear systems, switching signal estimation, state estimation, observer design, disturbance.

1. INTRODUCTION

The control and estimation problem for switched linear systems has attracted much attention over the past decade. On the one hand, both stabilization and tracking problem of switched systems have been discussed, e.g., in [1] and [2]. On the other hand, depending on the cases, various observability notions and observer design methods are already available in the existing literature. In particular, when the switching signal is known or can be measured while the states of the system are to be estimated, the observability has been studied from various perspectives, e.g., in [3], [4], [5], and [6]. The switching signal is determined by *switching time* (the set of time t when the *switching* occurs, or $\sigma(t)$ has discontinuity) and *active mode* (the integer value $\sigma(t)$ at time t). Furthermore, the asymptotic observer has been constructed for quite a general class of switched systems. For example, the observer proposed by [6] can estimate the states even when the system switches to unobservable subsystems as long as a certain accumulative observability holds for a certain time window. With the knowledge of switching signals, observability and observer design have also been extended to the nonlinear case by [7].

On the other hand, when the switching signal is not measurable and unknown, the estimation problem of both the switching signal and the states becomes more challenging, and many results are available under certain restrictions. In the case of discrete-time switched linear systems, [8], [9], [10], and [11] have studied observability, and [12] and [13] have proposed estimation algorithms in various situations depending on the existence of external inputs, process disturbance, and/or measurement noise. For continuous-time switched linear systems, [14], [15], and [16] have characterized observability for the state and the switching signal. Furthermore, for the estimation problem of the state and the switching signal, [17] has presented an estimation algorithm for the switching signal and the state, but the result of that research is limited to single-input single-output systems with structured perturbations. It also relies on a numerical estimation of the time derivatives of inputs and outputs to investigate the input-output behavior. [18] and [19] have considered the estimation of the switching signal for autonomous systems by the distribution theory, but the states are assumed to be known. In particular, [20] and [21] have proposed a hybrid observer to estimate both the switching signal and the states, which is composed of a location observer and

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a continuous (state) observer, but the active mode is not uniquely determined in [20] due to the existence of singular inputs, and no systematic construction of the location observer is given in [21].

In this paper, we propose a hybrid-type observer to estimate both switching signals and states for continuous-time switched linear systems. Emphasis is given to generality and implementability. In fact, the system may have external inputs, process disturbances, and/or measurement noises. The external inputs are assumed to be bounded (which may be reasonable in practice due to the input saturation). Moreover, we do not impose any additional assumptions on the properties of disturbances/noises except their boundedness. Because of them, immediate detection of switching time is not possible, and some delay of detection is unavoidable. The condition that the proposed design is based on characterizes the relationship between the amount of delay, the strength of the joint observability, and the sizes of states, inputs, disturbances and noises. The proposed design can be briefly described as follows. We run an observer from the initial time ceaselessly, while its internal states are updated from time to time by an 'estimation algorithm'. This estimation algorithm is employed to estimate the plant's active mode accurately and the plant's state approximately, which is inspired by the 'minimum distance criterion' proposed in [22]. When the algorithm finishes, we run another 'detection algorithm' to detect the switching within Δ delay, where Δ is our desired accuracy of detection.

This paper is organized as follows: Section 2 introduces the problem formulation and two implementable algorithms to estimate the active mode and the state and detect the switching time. Section 3 gives a numerical example to show the performance of our algorithm. Section 4 concludes the paper.

2. MAIN RESULTS

We consider a switched linear system given by

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + d(t),\\ y(t) &= C_{\sigma(t)}x(t) + n(t),\end{aligned} \tag{1}$$

where $x(t) \in R^{n_x}$ is the state, $u(t) \in R^{n_u}$ is the input, $y(t) \in R^{n_y}$ is the measured output, $d(t) \in R^{n_x}$ is the process disturbance, $n(t) \in R^{n_y}$ is the measurement noise, and $\sigma : [0,\infty) \mapsto \mathcal{N}$ is the switching signal with $\mathcal{N} := \{1,2,\cdots,N\}$. While each integer $1,2,\cdots,N$ is called *mode*, the switched system (1) at each mode (i.e., when $\sigma(t) = i \in \mathcal{N}$) becomes an LTI system, which we call a *subsystem*. Therefore, the switched system (1) consists of N subsystems (or modes). It is supposed that both the state $x(t)$ and the switching signal $\sigma(t)$ are unknown while $\sigma(\cdot)$ is assumed to be piecewise constant and right continuous. We assume that all subsystems of (1) are observable, i.e., the pair (A_i, C_i), $\forall i \in \mathcal{N}$, is observable. Moreover, we assume a stronger notion given as follows:

Joint Observability: For any $i, j \in \mathcal{N}$ such that $i \neq j$,

$$\mathrm{rank}\left[O_i^{(2n_x)}\ O_j^{(2n_x)}\right] = 2n_x$$

where $O_i^{(k)} = [C_i^\top\ (C_iA_i)^\top\ \cdots\ (C_iA_i^{k-1})^\top]^\top$.

Joint observability is equivalent to the fact that the extended system, whose system matrix is blockdiag(A_i, A_j) and its output matrix is $[C_i, C_j]$, is observable for every $i \neq j$ [14]. It is also equivalently said that the joint observability Gramian

$$W_{i,j}(t) := \int_0^t \begin{bmatrix}\phi_i^\top(\tau)\\ \phi_j^\top(\tau)\end{bmatrix}\begin{bmatrix}\phi_i(\tau) & \phi_j(\tau)\end{bmatrix} d\tau,$$

where $\phi_i(t) := C_i e^{A_i t}$, is nonsingular for any $t \neq 0$ and $i \neq j$. Therefore, with

$$\omega_{\min}(t) := \min_{i,j\in\mathcal{N}, i\neq j} \lambda_{\min}\left(W_{i,j}(t)\right),$$

where $\lambda_{\min}$ indicates the minimum eigenvalue, it is seen that $\omega_{\min}(t) > 0$ for $t > 0$.

Two modes $i, j \in \mathcal{N}$ with $i \neq j$ are distinguishable if and only if the joint observability holds [22, Lemma 1], [14, Lemma 1]. Thus, the joint observability is necessary for the active mode estimation, and this can be explained in the following way. Let $y_i(t;t_0,x_0,u)$ be the output at time t, which is generated from the initial condition x_0 at time t_0 under the input function $u(\cdot)$, no process disturbance, and no measurement noise, for the mode i. Then it can be shown that, for two modes $i, j \in \mathcal{N}$ and zero input,

$$\begin{aligned}&\int_0^t |y_i(\tau;0,x_0,0) - y_j(\tau;0,x_0',0)|^2 d\tau\\ &\quad = \begin{bmatrix} x_0^\top & -x_0'^\top \end{bmatrix} W_{i,j}(t) \begin{bmatrix} x_0\\ -x_0' \end{bmatrix}.\end{aligned} \tag{2}$$

It is clear that, if $\omega_{\min}(t) = 0$ with $t > 0$, then there are x_0 and x_0' (not both zero), and two different modes i and j such that $y_i(\tau;0,x_0,0) = y_j(\tau;0,x_0',0)$ for $0 \leq \tau \leq t$. Hence, the modes i and j cannot be distinguished from the output.

It is also noted that, with $x_0 = x_0'$ in (2), we can reduce (2) to

$$\int_0^t |y_i(\tau;0,x_0,0) - y_j(\tau;0,x_0,0)|^2 d\tau = x_0^\top \overline{W}_{i,j}(t) x_0,$$

where

$$\begin{aligned}\overline{W}_{i,j}(t) &:= \begin{bmatrix} I & -I \end{bmatrix} W_{i,j}(t) \begin{bmatrix} I\\ -I \end{bmatrix}\\ &= \int_0^t \left(\phi_i^\top(\tau) - \phi_j^\top(\tau)\right)\left(\phi_i(\tau) - \phi_j(\tau)\right) d\tau.\end{aligned}$$

If $W_{i,j}(t)$ is nonsingular, then $\overline{W}_{i,j}(t)$ is also nonsingular. Hence, with

$$\bar{\omega}_{\min}(t) := \min_{i,j\in\mathcal{N}, i\neq j} \lambda_{\min}\left(\overline{W}_{i,j}(t)\right),$$

we have $\bar{\omega}_{\min}(t) > 0$ for $t > 0$.

Finally, we present a condition that will enable the estimation of the active mode and the states even under process disturbances, measurement noises, and external inputs. For this purpose, let $\check{\mu} \geq 1$ and $\check{\lambda} \geq 0$ be such that $\|e^{A_i t}\| \leq \check{\mu} e^{\check{\lambda} t}$, $^\forall t \geq 0$, $^\forall i \in \mathcal{N}$. Note that $\|\cdot\|$ denotes the induced matrix 2-norm for a matrix and $|\cdot|$ denotes the Euclidean norm for a vector. In addition, let L_i be such that $(A_i - L_i C_i)$ is Hurwitz. Then, there exist $\hat{\mu} \geq 1$ and $\hat{\lambda} > 0$ such that $\|e^{(A_i - L_i C_i)t}\| \leq \hat{\mu} e^{-\hat{\lambda} t}$, $^\forall t \geq 0$, $^\forall i \in \mathcal{N}$. Define $L_{\max} := \max_{i\in\mathcal{N}} \|L_i\|$ and $C_{\max} := \max_{i\in\mathcal{N}} \|C_i\|$.

Assumption 1: The input u, the process disturbance d, and the measurement noise n are uniformly bounded, i.e.,

$$|u(t)| \leq u_{\max},\ |d(t)| \leq d_{\max},\ |n(t)| \leq n_{\max},\ ^\forall t \geq 0. \quad (3)$$

Moreover, there exist positive constants δ and Δ such that

$$\omega_{\min}(\delta)|x(t)|^2 > \left(u_{\max}\sqrt{N_u(\delta)} + 2(d_{\max}F_{\max}(\delta) + n_{\max})\sqrt{\delta}\right)^2, \quad (4)$$

$$\begin{aligned}\bar{\omega}_{\min}(\Delta)|x(t)|^2 &> \Big(u_{\max}\sqrt{N_u(\Delta)} + (d_{\max}F_{\max}(\Delta) + 2n_{\max})\sqrt{\Delta} \\ &\quad + C_{\max}\check{\mu} e^{\check{\lambda}\Delta}(2\check{\mu} e^{\check{\lambda}\Delta} + 2) \\ &\quad \times (d_{\max}\hat{E}_D(\delta) + n_{\max}\hat{E}_n(\delta))\sqrt{\Delta}\Big)^2, \quad (5)\end{aligned}$$

for all $t \geq 0$ where, with $h_i(t) := C_i e^{A_i t} B_i$ and $U_i(t) := \int_0^t \phi_i^\top(\tau)\phi_i(\tau)d\tau$ (observability Gramian for mode i),

$$N_u(\delta) := \max_{i,j\in\mathcal{N}, i\neq j} \int_0^\delta \left(\int_0^\tau \|h_i(s) - h_j(s)\| ds\right)^2 d\tau,$$

$$F_{\max}(\delta) := \max_{i\in\mathcal{N}} F_i(\delta) := \max_{i\in\mathcal{N}} \int_0^\delta \|\phi_i(\tau)\| d\tau,$$

$$M_{\max}(\delta) := \max_{i\in\mathcal{N}} M_i(\delta) := \max_{i\in\mathcal{N}} \int_0^\delta \|U_i^{-1}(\delta)\phi_i^\top(\tau)\| d\tau,$$

$$\hat{E}_d(\delta) := \hat{\mu} \max\left\{\max_{i\in\mathcal{N}} F_i(\delta)M_i(\delta), \frac{1}{\hat{\lambda}}\right\},$$

$$\hat{E}_D(\delta) := \hat{E}_d(\delta) + \frac{1}{\check{\lambda}}\left(1 - e^{-\check{\lambda}\Delta}\right),$$

$$\hat{E}_n(\delta) := \hat{\mu} \max\left\{M_{\max}(\delta), \frac{L_{\max}}{\hat{\lambda}}\right\}.$$

Remark 1: Joint observability is essential in Assumption 1 in the sense that (4) and (5) can never be satisfied if $\omega_{\min}(\delta) = 0$ and $\bar{\omega}_{\min}(\Delta) = 0$. □

Remark 2: To compute the minimum δ and Δ satisfying (4) and (5) for a given switched linear system may be difficult. Practically, one can first choose δ and Δ which will be seen to be related to the accuracy of the switching time estimation, and then compute the lower bound of $|x(t)|$ from (4) and (5) with the selected accuracy δ and Δ. This computation is done off-line. If the minimum norm of anticipated state trajectories for the given plant does not satisfy the computed lower bound, one can either compromise the accuracy by increasing δ or Δ to guarantee the switching estimation for all time, or sacrifice the persistency of estimation by maintaining δ and Δ to ensure the estimation precision only for large enough states. □

A few comments should be made on Assumption 1. When there is no process disturbance, measurement noise, nor input (so that $d_{\max} = n_{\max} = u_{\max} = 0$), the conditions (4) and (5) simply become $|x(t)| > 0$ because $\omega_{\min}(t) > 0$ and $\bar{\omega}_{\min}(t) > 0$ for any $t > 0$. Then this condition becomes necessary because, if $|x(t)| = 0$, then $x(t) = 0$ and $y(t) = 0$ for all time for any mode $i \in \mathcal{N}$, so that the estimation of active modes is not possible. Similarly, when the measurement noise is present, the state norm $|x(t)|$ should not be too small because a certain norm-bounded noise $n(t)$ may make the output $y(t) = C_i x(t) + n(t)$ identically zero while $x(t)$ is not. In this case, the estimation of the active mode or the state is not possible either. A similar observation can be made when the input u or the process disturbance d is present. Consider a jointly observable two-mode switched system $\Sigma_1 : \dot{x} = -x + u,\ y = x$ and $\Sigma_2 : \dot{x} = -2x + 2u,\ y = x$. With $u(t) \equiv 1$ and $x(0) = 1$, the output $y(t) = 1$, $^\forall t \geq 0$, for any switching signal, which disables the detection and estimation. Thus, conditions (4) and (5) can be understood as a way to avoid this pathological case. Moreover, the conditions illustrate the relationship among many factors for the detection and estimation to become easier, in the sense that the conditions are more likely to hold if $|x(t)|$ gets larger, if $u_{\max}$, $d_{\max}$, and $n_{\max}$ get smaller, or if $\omega_{\min}(\cdot)$ and $\bar{\omega}_{\min}(\cdot)$ get larger (i.e., joint observability gets stronger). Finally, it is observed that δ and Δ should not be too small because $\omega_{\min}(t)$ and $\bar{\omega}_{\min}(t)$ belong to $o(t)$ as $t \to 0$ (i.e., locally it has a higher order than t), which is clear from the fact that the right derivatives $\partial_+ \omega_{\min}(t)|_{t=0} = \lim_{t\to 0^+} \omega_{\min}(t)/t = 0$ and $\partial_+ \bar{\omega}_{\min}(t)|_{t=0} = \lim_{t\to 0^+} \bar{\omega}_{\min}(t)/t = 0$, while the right hand sides of (4) and (5) are linear around zero with respect to δ or Δ. This implies that when δ and Δ are very small, the left hand sides of (4) and (5) become smaller than the right hand sides.

Assumption 2: The switching has the dwell time of $\Delta + \delta + T_{\mathsf{comp}}$, and there is no switching for the initial time interval $[0, \delta + T_{\mathsf{comp}}]$ with some $T_{\mathsf{comp}} > 0$.

Remark 3: T_{comp} is an upper bound of T^j_{comp} which is a time consumed by the computation of an algorithm to be introduced. This will become clarified shortly. □

Based on the assumptions, the proposed estimation and detection algorithm is described as follows. Consider an observer

$$\dot{\hat{x}} = (A_{\hat{\sigma}} - L_{\hat{\sigma}} C_{\hat{\sigma}})\hat{x} + B_{\hat{\sigma}} u + L_{\hat{\sigma}} y, \quad \dot{\hat{\sigma}} = 0, \tag{6}$$

where $\hat{x}(t)$ is the estimate of the plant state $x(t)$ and $\hat{\sigma}(t)$ is the estimate of the switching signal $\sigma(t)$. This observer runs continually from the initial time, but its states $\hat{x}(t)$ and $\hat{\sigma}(t)$ are updated from time to time by another computation algorithm (called Algorithm EST). The occurrence of switching is detected by an Algorithm DET (to be introduced). Note that "EST" is short for estimation and "DET" is short for detection. We denote by $\hat{t}_j$ the estimated j-th switching time. For convenience, we let $\hat{t}_0 = 0$.

Operation of the algorithms: As an initialization, let $j = 0$ and start the observer (6) with arbitrary $\hat{x}(0)$ and $\hat{\sigma}(0)$. At time $\hat{t}_j$ (which is 0 when $j = 0$), call Algorithm EST. This algorithm completes soon after the time $t = \hat{t}_j + \delta$ when it updates $\hat{x}$ and $\hat{\sigma}$ of (6). After the completion of Algorithm EST (i.e., at $t = \hat{t}_j + \delta + T^j_{\mathsf{comp}}$ with $0 < T^j_{\mathsf{comp}} \le T_{\mathsf{comp}}$), Algorithm DET, which monitors the occurrence of switching, starts. If switching occurs, Algorithm DET detects it soon and reports the switching. At this moment, increase j, mark the time as $\hat{t}_j$, and repeat (i.e., start Algorithm EST because the time is $t = \hat{t}_j$). Note that these algorithms are performed as a background operation of the computer, and observer (6) runs continuously to generate $\hat{x}(t)$ and $\hat{\sigma}(t)$.

The role of Algorithm EST is to determine the active mode σ precisely and estimate the plant state x under the assumption that there is no switching while Algorithm EST runs (which is satisfied by Assumption 2). If there is neither process disturbance d nor measurement noise n, then the estimation of the plant state also becomes exact. According to the operation scheme above, this algorithm is called at every $\hat{t}_j$ and continues until $\hat{t}_j + \delta + T^j_{\mathsf{comp}}$.

Algorithm EST

Require: $\check{x}_i(\hat{t}_j) \leftarrow 0$, $z_i(\hat{t}_j) \leftarrow 0$ for all $i \in \mathcal{N}$

1: **while** $t < \hat{t}_j + \delta$ **do**

2: Integrate the following system for all $i \in \mathcal{N}$

$$\dot{\check{x}}_i = A_i \check{x}_i + B_i u,$$
$$\dot{z}_i = -A_i^\top z_i + C_i^\top (y - C_i \check{x}_i).$$

3: **end while**

4: **for** $i = 1$ to N **do**

5: $X_i^* \leftarrow U_i^{-1}(\delta) e^{A_i^\top \delta} z_i(\hat{t}_j + \delta)$

6: Run the following system from $\hat{t}_j$ to $\hat{t}_j + \delta$

$$\dot{\check{x}}_i = A_i \check{x}_i + B_i u, \qquad \check{x}_i(\hat{t}_j) = X_i^*,$$
$$\dot{s}_i = |y - C_i \check{x}_i|^2, \qquad s_i(\hat{t}_j) = 0.$$

7: $I_i \leftarrow s_i(\hat{t}_j + \delta)$

8: **end for**

9: $k \leftarrow \arg\min_{i \in \mathcal{N}} I_i$

10: Run $\dot{\xi} = (A_k - L_k C_k)\xi + B_k u + L_k y$ from $\hat{t}_j$ to the current time t with $\xi(\hat{t}_j) = X_k^*$.

11: $\hat{\sigma}(t) \leftarrow k$, $\hat{x}(t) \leftarrow \xi(t)$ where $t = \hat{t}_j + \delta + T^j_{\mathsf{comp}}$.

'Run' in lines 6 and 10 means that the numerical integration is performed for the given time period, while 'Integrate' of line 2 implies that the integration is in pace with the real time t. The incoming data u and y are stored in the memory queue from the time $\hat{t}_j$ for the use in lines 6 and 10. Here T^j_{comp} is the time consumed by lines 4–11, which is determined on-line. We assume that the computation is fast enough and $T^j_{\mathsf{comp}} \le T_{\mathsf{comp}}$ for all j. By the completion of the algorithm, the internal states $\hat{x}$ and $\hat{\sigma}$ of (6) are updated, which satisfies the following property.

Lemma 1: Under assumptions (3) and (4), if there is no switching from $\hat{t}_j$ to $\hat{t}_j + \delta + T^j_{\mathsf{comp}}$, then Algorithm EST (with the observer (6)) guarantees that,

$$\hat{\sigma}(t) = \sigma(t) \tag{7}$$
$$|x(t) - \hat{x}(t)| \le d_{\max} \hat{E}_d(\delta) + n_{\max} \hat{E}_n(\delta) \tag{8}$$

for all $t \ge \hat{t}_j + \delta + T^j_{\mathsf{comp}}$ until the next switching occurs.

Proof: This proof is mostly inspired by the 'minimum distance criterion' proposed in [22]. By solving the linear differential equation of line 2 in Algorithm EST, it is seen from line 5 that

$$\begin{aligned} X_i^* = U_i^{-1}(\delta) \int_{\hat{t}_j}^{\hat{t}_j+\delta} & \phi_i^\top (s - \hat{t}_j) \\ & \times \left(y(s) - \int_{\hat{t}_j}^{s} h_i(s-\tau) u(\tau) d\tau \right) ds. \end{aligned} \tag{9}$$

Then it follows that the above X_i^* minimizes

$$J_i(X_i) := \int_{\hat{t}_j}^{\hat{t}_j+\delta} |y(t) - y_i(t; \hat{t}_j, X_i, u)|^2 dt,$$

which can be shown by differentiating $J_i(X_i)$ with respect to X_i. That is, we obtain X_i^* by solving the following equation:

$$\begin{aligned} \frac{d}{dX_i} J_i(X_i) &= \frac{d}{dX_i} \int_{\hat{t}_j}^{\hat{t}_j+\delta} |y(t) - y_i(t; \hat{t}_j, X_i, u)|^2 dt \\ &= \int_{\hat{t}_j}^{\hat{t}_j+\delta} \frac{\partial}{\partial X_i} |y(t) - y_i(t; \hat{t}_j, X_i, u)|^2 dt \\ &= \int_{\hat{t}_j}^{\hat{t}_j+\delta} \frac{\partial}{\partial X_i} (\zeta(t) - \phi_i(t - \hat{t}_j) X_i)^\top (\zeta(t) - \phi_i(t - \hat{t}_j) X_i) dt \\ &= -2 \int_{\hat{t}_j}^{\hat{t}_j+\delta} \phi_i^\top (t - \hat{t}_j)(\zeta(t) - \phi_i(t - \hat{t}_j) X_i) dt = 0, \end{aligned}$$

where $\zeta(t) := y(t) - \int_{\hat{t}_j}^{t} h_i(t-\tau) u(\tau) d\tau$.

It is also seen that $J_i(X_i^*)$ is computed by lines 6 and 7 (i.e., $J_i(X_i^*) = I_i$). Finally, since $y(s) = \phi_\sigma(s - \hat{t}_j) x(\hat{t}_j) +$

$\int_{\hat{t}_j}^{s} h_\sigma(s-\tau)u(\tau)d\tau + v(s;\hat{t}_j)$ where $v(s;\hat{t}_j) := \int_{\hat{t}_j}^{s} \phi_\sigma(s-\tau)d(\tau)d\tau + n(s)$, it follows from (9) that

$$X_\sigma^* = x(\hat{t}_j) + U_\sigma^{-1}(\delta)\int_{\hat{t}_j}^{\hat{t}_j+\delta} \phi_\sigma^\top(s-\hat{t}_j)v(s;\hat{t}_j)ds. \quad (10)$$

On the other hand, by a routine computation (see the Appendix), it can be shown that

$$\sqrt{J_\sigma(X_\sigma^*)} \le (d_{\max}F_{\max}(\delta) + n_{\max})\sqrt{\delta} \quad (11)$$

and that

$$\begin{aligned}\sqrt{J_i(X_i^*)} \ge{}& \sqrt{\omega_{\min}(\delta)}|x(\hat{t}_j)| - u_{\max}\sqrt{N_u(\delta)} \\ &- (d_{\max}F_{\max}(\delta) + n_{\max})\sqrt{\delta}\end{aligned} \quad (12)$$

for all $i \neq \sigma$. Then, by (4), the algorithm in line 9 yields the correct active mode σ.

Now, to show (8), let $\varepsilon := x - \xi$, which obeys $\dot{\varepsilon} = (A_\sigma - L_\sigma C_\sigma)\varepsilon + d - L_\sigma n$ with the initial condition $\varepsilon(\hat{t}_j) = x(\hat{t}_j) - X_\sigma^*$ satisfying $|\varepsilon(\hat{t}_j)| \le M_\sigma(\delta)(d_{\max}F_\sigma(\delta) + n_{\max})$ from (10). Then we have that

$$\begin{aligned}|\varepsilon(t)| &= \Big| e^{(A_\sigma - L_\sigma C_\sigma)(t-\hat{t}_j)}\varepsilon(\hat{t}_j) \\ &\qquad + \int_{\hat{t}_j}^{t} e^{(A_\sigma - L_\sigma C_\sigma)(t-\tau)}(d(\tau) - L_\sigma n(\tau))d\tau \Big| \\ &\le \hat{\mu}e^{-\hat{\lambda}(t-\hat{t}_j)}|\varepsilon(\hat{t}_j)| \\ &\qquad + \int_{\hat{t}_j}^{t} \hat{\mu}e^{-\hat{\lambda}(t-\tau)}d\tau(d_{\max} + L_{\max}n_{\max}) \\ &\le \hat{\mu}e^{-\hat{\lambda}(t-\hat{t}_j)}M_\sigma(\delta)(d_{\max}F_\sigma(\delta) + n_{\max}) \\ &\qquad + \frac{\hat{\mu}}{\hat{\lambda}}(1 - e^{-\hat{\lambda}(t-\hat{t}_j)})(d_{\max} + L_{\max}n_{\max}) \\ &\le d_{\max}\hat{\mu}\max\left\{\max_{i\in\mathcal{N}} F_i(\delta)M_i(\delta), 1/\hat{\lambda}\right\} \\ &\qquad + n_{\max}\hat{\mu}\max\left\{M_{\max}(\delta), L_{\max}/\hat{\lambda}\right\} \\ &= d_{\max}\hat{E}_d(\delta) + n_{\max}\hat{E}_n(\delta)\end{aligned} \quad (13)$$

for $t \ge \hat{t}_j$. By noting that the dynamics in line 10 are the same as (6), it is seen that (8) holds for all future time from $\hat{t}_j + \delta + T_{\mathsf{comp}}^j$ as long as $\hat{\sigma}(t) = \sigma(t)$. □

The role of Algorithm DET is to detect switching time by monitoring the output of the plant. It is called immediately after the completion of Algorithm EST, i.e., at $t = \hat{t}_j + \delta + T_{\mathsf{comp}}^j$ which we denote by $\bar{t}_j$ from now on. Note that $\hat{\sigma}(\bar{t}_j)$ is the same as the true active mode $\sigma(\bar{t}_j)$, and $\hat{x}(\bar{t}_j)$ satisfies (8) by the completion of Algorithm EST. Let $S_\Delta := (n_{\max} + C_{\max}\check{\mu}e^{\check{\lambda}\Delta}(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n))^2\Delta$, and pick $s_{\max} > S_\Delta$. (From now on, $\hat{E}_d$ and $\hat{E}_D$ represent $\hat{E}_d(\delta)$ and $\hat{E}_D(\delta)$ given in Assumption 1 for simple notation.)

Algorithm DET

Require: $\check{x}(t) \leftarrow \hat{x}(t)$, $T(t) \leftarrow 0$, $s(t) \leftarrow 0$, $\mathsf{flag} \leftarrow 0$, $t_{\mathsf{latch}} \leftarrow 0$ where t is the current time.

1: **repeat**

2: Integrate

$$\dot{\check{x}} = A_{\hat{\sigma}}\check{x} + B_{\hat{\sigma}}u, \quad (14)$$
$$\dot{s} = |y - C_{\hat{\sigma}}\check{x}|^2, \quad (15)$$
$$\dot{T} = 1. \quad (16)$$

3: **if** $T(t) \ge \Delta$ **then**

$$\check{x}(t) \leftarrow \hat{x}(t), \quad T(t) \leftarrow 0 \quad (17)$$

4: **end if**

5: **if** $s(t) \ge s_{\max}$ **then**

6: $\mathsf{flag} \leftarrow 1$, $t_{\mathsf{latch}} \leftarrow t$, $s(t_{\mathsf{latch}}) \leftarrow 0$

7: **end if**

8: **if** $t - t_{\mathsf{latch}} \ge \Delta$ **then**

9: $\mathsf{flag} \leftarrow 0$

10: **end if**

11: **until** either one of the following holds:

$$|\check{x}(t) - \hat{x}(t)| > (\check{\mu}e^{\check{\lambda}\Delta} + 1)(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n), \quad (18)$$

$$\text{or} \quad \mathsf{flag}\cdot s_{\max} + s(t) - s(t-\Delta) > S_\Delta. \quad (19)$$

12: $j \leftarrow j+1$, mark the estimated switching time $\hat{t}_j \leftarrow t$.

Remark 4: When (19) is evaluated, we consider that $s(t) = 0$ for $t < \bar{t}_j$. □

Remark 5: In the algorithm, lines 5–10 and the term $\mathsf{flag}\cdot s_{\max}$ in (19) is devised, for practical reasons, to avoid the ever-increasing $s(t)$ from (15) when there is no more switching. Therefore, the analysis is equivalently completed without lines 5–10 and with

$$s(t) - s(t-\Delta) > S_\Delta \quad (20)$$

instead of (19). □

We now prove that Algorithm DET detects the switching within the maximum delay of Δ. For this purpose, suppose that a switching occurs at $t^* > \hat{t}_j + \delta + T_{\mathsf{comp}}^j = \bar{t}_j$, and let $\check{e} := x - \check{x}$ and $\hat{e} := x - \hat{x}$. Then, while $\hat{\sigma}(t) = \sigma(t)$ for the interval $[\bar{t}_j, t^*)$, we have that $|\hat{e}(t)| \le d_{\max}\hat{E}_d + n_{\max}\hat{E}_n$ by Lemma 1, and $\check{e}(t)$ obeys $\dot{\check{e}} = A_\sigma\check{e} + d$. Since $\check{e}(\bar{t}_j) = \hat{e}(\bar{t}_j)$ and $\check{e}(t)$ is reset to $\hat{e}(t)$ periodically with the period Δ by (17), we have that

$$\begin{aligned}&|\check{e}(t)| \\ &= \left| e^{A_\sigma(t-\bar{t}_j-k\Delta)}\check{e}(\bar{t}_j + k\Delta) + \int_{\bar{t}_j+k\Delta}^{t} e^{A_\sigma(t-\tau)}d(\tau)d\tau \right| \\ &\le \check{\mu}e^{\check{\lambda}\Delta}(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n)\end{aligned}$$

for $\bar{t}_j \le t \le t^*$ where $\bar{t}_j + k\Delta$ is the last reset time before time t. This in turn implies that

$$\begin{aligned}|\check{x}(t) - \hat{x}(t)| &\le |\check{e}(t)| + |\hat{e}(t)| \\ &\le (\check{\mu}e^{\check{\lambda}\Delta} + 1)(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n)\end{aligned} \quad (21)$$

for $\bar{t}_j \le t \le t^*$. Note that $\hat{E}_d \le \hat{E}_D$ is used in (21).

Lemma 2: The conditions (18) and (19) (or (20)) are never activated during the interval $[\bar{t}_j, t^*]$.

Proof: It is clear from (21) that (18) is not activated for the interval. For (20), we note that

$$
\begin{aligned}
&(s(t)-s(t-\Delta))^{\frac{1}{2}} \\
&= \left(\int_{\max\{\bar{t}_j, t-\Delta\}}^{t} |y(\tau) - C_{\hat{\sigma}}\check{x}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&= \left(\int_{\max\{\bar{t}_j, t-\Delta\}}^{t} |C_{\sigma}x(\tau) + n(\tau) - C_{\hat{\sigma}}\check{x}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&= \left(\int_{\max\{\bar{t}_j, t-\Delta\}}^{t} |n(\tau) + C_{\hat{\sigma}}\check{e}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&\le \left(\int_{\max\{\bar{t}_j, t-\Delta\}}^{t} |n(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\max\{\bar{t}_j, t-\Delta\}}^{t} |C_{\hat{\sigma}}\check{e}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&\le n_{\max}\sqrt{\Delta} + C_{\max}\check{\mu}e^{\check{\lambda}\Delta}(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n)\sqrt{\Delta} \\
&= \sqrt{S_\Delta}
\end{aligned}
$$

as long as $\hat{\sigma}(t) = \sigma(t)$, which proves the claim. □

Lemma 3: Under Assumptions 1 and 2, within Δ after the switching time t^*, either (18) or (19) is activated, so that Algorithm DET completes at time $t = \hat{t}_{j+1} \le t^* + \Delta$.

Proof: If (18) becomes active in the interval $[t^*, t^*+\Delta]$, the proof is complete. Hence, assuming that

$$
|\check{x}(t) - \hat{x}(t)| \le (\check{\mu}e^{\check{\lambda}\Delta} + 1)(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n), \tag{22}
$$

for $t^* \le t \le t^* + \Delta$, we show that (20) becomes active in $[t^*, t^*+\Delta]$. Since $(s(t^*) - s(t^*-\Delta)) \le S_\Delta$ by Lemma 2 and $(s(t) - s(t-\Delta))$ is continuous, it is enough to show that $(s(t^*+\Delta) - s(t^*)) > S_\Delta$, which implies activation of (19) in the interval $[t^*, t^*+\Delta]$.

Let us define a virtual error variable

$$
\tilde{e}_i(t) := \check{x}(t) - \left(e^{A_i(t-t^*)}x(t^*) + \int_{t^*}^{t} e^{A_i(t-s)}B_i u(s)ds\right)
$$

for $t \ge t^*$, where $\check{x}(t)$ is the solution of (14) while the parenthesis is the solution of the plant (1) from $x(t^*)$ at $t = t^*$ as if the mode is i and $d(t) = 0$ for $t \ge t^*$. Note that $\check{x}(t)$ obeys (14) with $\hat{\sigma}(t) = \hat{\sigma}(\bar{t}_j)$ for $t \ge t^* > \bar{t}_j$ (in spite of the switching at $t = t^*$). We claim that

$$
|\tilde{e}_{\hat{\sigma}}(t)| \le \check{\mu}e^{\check{\lambda}\Delta}(2\check{\mu}e^{\check{\lambda}\Delta} + 1)(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n),
$$

for $t^* \le t \le t^* + \Delta$. To see this, we consider two cases (keeping in mind that $\check{x}(t)$ is reset to $\hat{x}(t)$ every Δ period). Let t_{u} be the time of the update (17) between t^* and $t^*+\Delta$. If $t_{\mathrm{u}} = t^*$ (and another update at $t^*+\Delta$), then

$$
\begin{aligned}
|\tilde{e}_{\hat{\sigma}}(t)| &= |e^{A_{\hat{\sigma}}(t-t^*)}(\check{x}(t^*) - x(t^*))| \\
&\le \check{\mu}e^{\check{\lambda}\Delta}(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n),
\end{aligned}
$$

which proves the claim. Now suppose that $t_{\mathrm{u}} \in (t^*, t^*+\Delta)$. For $t^* \le t < t_{\mathrm{u}}$, we have $\tilde{e}_{\hat{\sigma}}(t) = e^{A_{\hat{\sigma}}(t-t^*)}(\check{x}(t^*) - x(t^*))$. For $t_{\mathrm{u}} \le t \le t^*+\Delta$, we have by linearity that

$$
\begin{aligned}
\tilde{e}_{\hat{\sigma}}(t) &= e^{A_{\hat{\sigma}}(t-t_{\mathrm{u}})}(\check{x}(t_{\mathrm{u}}) - \check{x}(t_{\mathrm{u}}^-) + \tilde{e}_{\hat{\sigma}}(t_{\mathrm{u}}^-)) \\
&= e^{A_{\hat{\sigma}}(t-t_{\mathrm{u}})}(\hat{x}(t_{\mathrm{u}}) - \check{x}(t_{\mathrm{u}}^-)) \\
&\quad + e^{A_{\hat{\sigma}}(t-t^*)}(\check{x}(t^*) - x(t^*)).
\end{aligned} \tag{23}
$$

We note that, by (22), $|\hat{x}(t_{\mathrm{u}}) - \check{x}(t_{\mathrm{u}}^-)| \le (\check{\mu}e^{\check{\lambda}\Delta} + 1)(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n)$, and that $|\check{x}(t^*) - x(t^*)| \le \check{\mu}e^{\check{\lambda}\Delta}(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n)$, for $t^* \le t \le t^*+\Delta$. Combining them, the claim follows.

Finally,

$$
\begin{aligned}
&\left(s(t^*+\Delta) - s(t^*)\right)^{\frac{1}{2}} \\
&= \left(\int_{t^*}^{t^*+\Delta} |y(\tau) - C_{\hat{\sigma}}\check{x}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&= \left(\int_{t^*}^{t^*+\Delta} \Big|\big(\phi_\sigma(\tau-t^*) - \phi_{\hat{\sigma}}(\tau-t^*)\big)x(t^*) + v(\tau;t^*)\right. \\
&\quad \left. + \int_{t^*}^{\tau}(h_\sigma(\tau-s) - h_{\hat{\sigma}}(\tau-s))u(s)ds - C_{\hat{\sigma}}\tilde{e}_{\hat{\sigma}}(\tau)\Big|^2 d\tau\right)^{\frac{1}{2}} \\
&\ge \left(\int_{t^*}^{t^*+\Delta} \left|\big(\phi_\sigma(\tau-t^*) - \phi_{\hat{\sigma}}(\tau-t^*)\big)x(t^*)\right|^2 d\tau\right)^{\frac{1}{2}} \\
&\quad - \left(\int_{t^*}^{t^*+\Delta}\left|\int_{t^*}^{\tau}\big(h_\sigma(\tau-s) - h_{\hat{\sigma}}(\tau-s)\big)u(s)ds\right|^2 d\tau\right)^{\frac{1}{2}} \\
&\quad - \left(\int_{t^*}^{t^*+\Delta}|v(\tau;t^*)|^2 d\tau\right)^{\frac{1}{2}} - \left(\int_{t^*}^{t^*+\Delta}|C_{\hat{\sigma}}\tilde{e}_{\hat{\sigma}}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&\ge \left(\lambda_{\min}(\overline{W}_{\sigma,\hat{\sigma}}(\Delta))\,|x(t^*)|^2\right)^{\frac{1}{2}} \\
&\quad - \left(\int_0^{\Delta}\left(\int_0^{\tau}\|h_\sigma(\tau-s) - h_{\hat{\sigma}}(\tau-s)\|ds\right)^2 d\tau\, u_{\max}^2\right)^{\frac{1}{2}} \\
&\quad - \left(\int_{t^*}^{t^*+\Delta}|v(\tau;t^*)|^2 d\tau\right)^{\frac{1}{2}} - \left(\int_{t^*}^{t^*+\Delta}|C_{\hat{\sigma}}\tilde{e}_{\hat{\sigma}}(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&\ge \sqrt{\bar{\omega}_{\min}(\Delta)}|x(t^*)| \\
&\quad - u_{\max}\sqrt{N_u(\Delta)} - (d_{\max}F_{\max}(\Delta) + n_{\max})\sqrt{\Delta} \\
&\quad - C_{\max}\check{\mu}e^{\check{\lambda}\Delta}(2\check{\mu}e^{\check{\lambda}\Delta} + 1)(d_{\max}\hat{E}_D + n_{\max}\hat{E}_n)\sqrt{\Delta}.
\end{aligned}
$$

By (5), we have $s(t^*+\Delta) - s(t^*) > S_\Delta$, which completes the proof. □

Combining Lemmas 1, 2, and 3 with Assumption 2, the analysis leads to the following summary.

Theorem 1: Under Assumptions 1 and 2, the observer (6) with the proposed scheme employing Algorithms EST and DET guarantees

- detection of the switching time by $\hat{t}_j$ within the delay of Δ,
- estimation of the active mode by $\hat{\sigma}$ within the delay of $\Delta+\delta+T_{\mathrm{comp}}$,
- approximate estimation of the plant state by $\hat{x}$ within the error of $d_{\max}\hat{E}_d(\delta)+n_{\max}\hat{E}_n(\delta)$ (see (8)) for each time interval from $\hat{t}_j+\delta+T^j_{\mathrm{comp}}$, $j\geq 0$, to the next switching time.

Remark 6: Note that there is a blackout period from the actual switching time to the next $\hat{t}_j+\delta+T^j_{\mathrm{comp}}$, on which the state estimate $\hat{x}(t)$ and the switching signal estimate $\hat{\sigma}(t)$ of the observer (6) are not very meaningful. Algorithm EST can return the estimates $\hat{x}$ and $\hat{\sigma}$ for the interval $[\hat{t}_j, \hat{t}_j+\delta+T^j_{\mathrm{comp}}]$ at its completion time, which is however not a real-time estimation.

3. A NUMERICAL EXAMPLE

Many practical plants such as undamped oscillators have the following dynamics

$$\ddot{y}(t)+\rho_i^2 y(t)=u(t)$$

which has a state-space representation of the form:

$$\dot{x}(t)=\begin{bmatrix}0 & \rho_i\\ -\rho_i & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ \frac{1}{\rho_i}\end{bmatrix}u(t), \qquad (24)$$
$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}x(t).$$

Consider a two-mode switched linear system (1) whose subsystems have the form of (24) with $\rho_1=1$, $\rho_2=2$, $x(t)=[x_1(t)\ x_2(t)]^\top$, $u(t)=1$, $d(t)=10^{-3}\sin(t)$, $n(t)=10^{-3}\sin(3t)$ and unknown switching signal $\sigma(t)=2$ when $t\in[0,2)\cup[10,12)$ and $\sigma(t)=1$ when $t\in[2,10)\cup[12,20]$. Choose design parameters by $\delta=0.3, \Delta=0.25$. Therefore, Assumption 2 is satisfied (with some $T_{\mathrm{comp}}>0$). Moreover, with initial state $x(0)=[10\ 10]^\top$, the state response is shown in Fig. 1, which implies Assumption 1 is satisfied because $|x(t)|>12.837$ for all t. Thus, we can use the proposed Algorithms EST and DET to estimate the switching signal and the state.

By simulation, the estimated active mode sequence is as follows: $\{2,1,2,1\}$. The estimated switching time with Δ precision is as follows: $\hat{t}_1=2.058$, $\hat{t}_2=10.013$, and $\hat{t}_3=12.026$. The estimated state is shown in Fig. 2. From the simulation results, we conclude that the switching signal can be well detected with short time delay. Moreover, a good state estimation is achieved except for short initial time interval and short time interval after switching.

The condition of Assumption 1 is conservative one. For example, with a solution $x(t)$ such that $x(0)=0.01\times[10\ 10]^\top$, the bounds in Assumption 1 are not satisfied (because $|x(t)|$ is too small). Even in this case, the simulation results show that the correct active mode is estimated with the estimated switching time as follows: $\hat{t}_1=2.209$, $\hat{t}_2=10.117$, and $\hat{t}_3=12.230$.

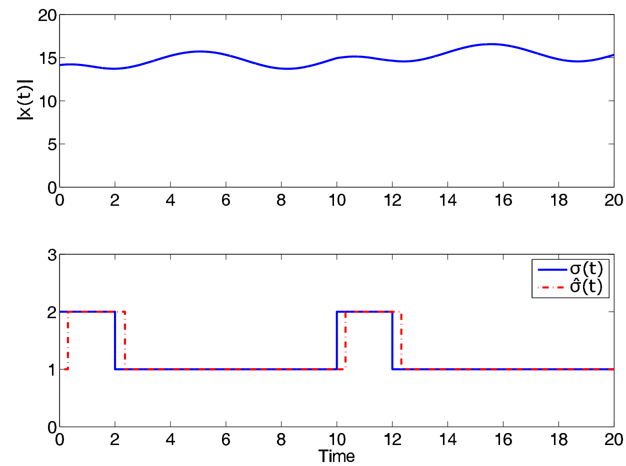

Fig. 1. Plot of $|x(t)|$ and the switching signal estimation.

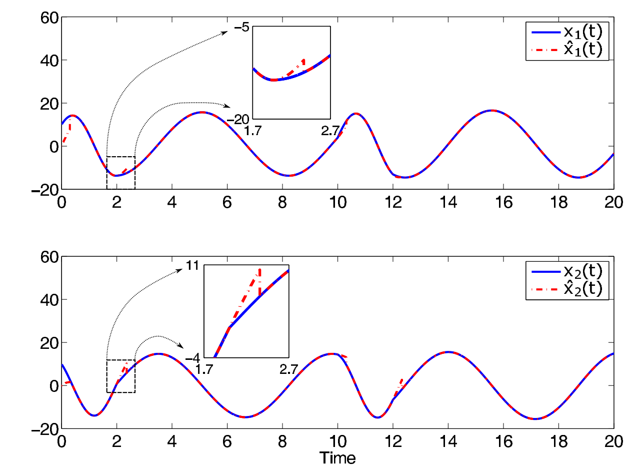

Fig. 2. State estimation for x_1 and x_2.

4. CONCLUSION

In this paper, we have proposed a robust estimation algorithm for the switching signal and the state of continuous-time switched linear systems under process disturbances and measurement noises. This research makes the following contributions. Firstly, inspired by the work of [22], an active mode estimation algorithm is proposed to tackle the bounded process disturbance, measurement noise, and external input. Secondly, a detection algorithm is presented to detect switching with Δ time precision, which involves some periodic updating strategy of observers and threshold conditions. Thirdly, the state estimation is performed ceaselessly except for the relatively short blackout time. The simulation results have demonstrated the effectiveness of the proposed algorithm.

APPENDIX A

A.1. Derivation of (11) and (12)

The detailed computation omitted in the proof of Lemma 1 is as follows. With $w_\sigma(\hat{t}_j) := U_\sigma^{-1}(\delta)\int_{\hat{t}_j}^{\hat{t}_j+\delta}\phi_\sigma^\top(s-\hat{t}_j)v(s;\hat{t}_j)ds$, it can be shown that

$$\begin{aligned}
\sqrt{J_\sigma(X_\sigma^*)} &= \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}|y(t)-y_\sigma(t;\hat{t}_j,X_\sigma^*,u)|^2dt\right)^{\frac{1}{2}}\\
&= \Bigg(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\big|y_\sigma(t;\hat{t}_j,x(\hat{t}_j),u)+v(t;\hat{t}_j)\\
&\qquad -y_\sigma(t;\hat{t}_j,x(\hat{t}_j)+w_\sigma(\hat{t}_j),u)\big|^2dt\Bigg)^{\frac{1}{2}}\\
&= \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}|v(t;\hat{t}_j)-y_\sigma(t;\hat{t}_j,w_\sigma(\hat{t}_j),0)|^2dt\right)^{\frac{1}{2}}\\
&= \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}v^\top(t;\hat{t}_j)v(t;\hat{t}_j)dt-w_\sigma^\top(\hat{t}_j)U_\sigma(\delta)w_\sigma(\hat{t}_j)\right)^{\frac{1}{2}}\\
&\le \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}v^\top(t;\hat{t}_j)v(t;\hat{t}_j)dt\right)^{\frac{1}{2}}\\
&\le \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\left|\int_{\hat{t}_j}^{t}\phi_\sigma(t-\tau)d(\tau)d\tau\right|^2dt\right)^{\frac{1}{2}}\\
&\quad +\left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}|n(t)|^2dt\right)^{\frac{1}{2}}\\
&\le d_{\max}\left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\left(\int_{\hat{t}_j}^{t}\|\phi_\sigma(t-\tau)\|d\tau\right)^2dt\right)^{\frac{1}{2}}+n_{\max}\sqrt{\delta}\\
&\le d_{\max}\left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\left(\int_{0}^{t-\hat{t}_j}\|\phi_\sigma(s)\|ds\right)^2dt\right)^{\frac{1}{2}}+n_{\max}\sqrt{\delta}\\
&\le d_{\max}\left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\left(\int_{0}^{\delta}\|\phi_\sigma(s)\|ds\right)^2dt\right)^{\frac{1}{2}}+n_{\max}\sqrt{\delta}\\
&\le (d_{\max}F_{\max}(\delta)+n_{\max})\sqrt{\delta}
\end{aligned}$$

and that for all $i\neq\sigma$

$$\begin{aligned}
\sqrt{J_i(X_i^*)} &= \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}|y(t)-y_i(t;\hat{t}_j,X_i^*,u)|^2dt\right)^{\frac{1}{2}}\\
&= \left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\big|y_\sigma(t;\hat{t}_j,x(\hat{t}_j),u)+v(t;\hat{t}_j)-y_i(t;\hat{t}_j,X_i^*,u)\big|^2dt\right)^{\frac{1}{2}}\\
&= \Bigg(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\Big|(y_\sigma(t;\hat{t}_j,x(\hat{t}_j),0)-y_i(t;\hat{t}_j,X_i^*,0))\\
&\qquad +(y_\sigma(t;\hat{t}_j,0,u)-y_i(t;\hat{t}_j,0,u))+v(t;\hat{t}_j)\Big|^2dt\Bigg)^{\frac{1}{2}}\\
&\ge \left(\lambda_{\min}(W_{\sigma,i}(\delta))|x(\hat{t}_j)|^2\right)^{\frac{1}{2}}\\
&\quad -\left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}\left|\int_{\hat{t}_j}^{t}(h_\sigma(t-\tau)-h_i(t-\tau))u(\tau)d\tau\right|^2dt\right)^{\frac{1}{2}}\\
&\quad -\left(\int_{\hat{t}_j}^{\hat{t}_j+\delta}v^\top(t;\hat{t}_j)v(t;\hat{t}_j)dt\right)^{\frac{1}{2}}\\
&\ge \sqrt{\omega_{\min}(\delta)}|x(\hat{t}_j)|-u_{\max}\sqrt{N_u(\delta)}\\
&\quad -(d_{\max}F_{\max}(\delta)+n_{\max})\sqrt{\delta}.
\end{aligned}$$

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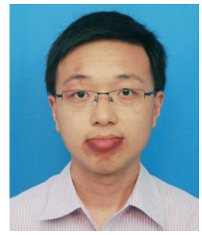

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